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BRIEF COMMUNICATION

ON SOO'S EQUATIONS FOR THE ONE-DIMENSIONAL MOTION OF SINGLE-COMPONENT TWO-PHASE FLOWS†

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In a recent article, Soo (1976) presents a set of equations for the one-dimensional adiabatic motion of two-phase single-component flow. The purpose of this note is to examine the validity of these equations in a specific application and offer an explanation for the resulting inconsistencies.

The continuity equation for one phase of a two-phase mixture is presented by Soo as

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x}(\rho_1 u_1) = \Gamma_1 \qquad [1]$$

where ρ_1 is the bulk density (mass per unit volume), u_1 the mean velocity and Γ_1 the generation rate of phase 1 per unit volume of mixture. The continuity equation for the second phase is the same as [1] with the subscripts replaced by 2 in all cases. Addition of the continuity equations yields the mixture continuity equation:

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial}{\partial x}(\rho_m u_m) = 0 \qquad [2]$$

where the mixture density and velocity are defined by

$$\rho_m = \rho_1 + \rho_2 \qquad [3a]$$

and

$$\rho_m u_m = \rho_1 u_1 + \rho_2 u_2. \qquad [3b]$$

The summation of the generation terms gives zero, since Γ_1 is equal to $-\Gamma_2$.

The momentum equation for phase 1 is presented as

$$\rho_1 \frac{\partial u_1}{\partial t} + \rho_1 u_1 \frac{\partial u_1}{\partial x} = -\phi_1 \frac{\partial P}{\partial x} + I_{12} + V_{12} \qquad [4]$$

where ϕ_1 is the volume fraction of phase 1 and P is the system pressure. The term I_{12} is defined as:

$$I_{12} = \frac{-\rho_2}{\rho_1 + \rho_2}(u_1 - u_2)\Gamma_1 + \frac{\partial}{\partial x}\left[\frac{\rho_1 \rho_2^2}{(\rho_1 + \rho_2)^2}(u_1 - u_2)^2\right] \qquad [5]$$

and identified as the "inertial interaction" force per unit volume of mixture. The last term, V_{12},

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†This work was performed during the author's sabbatical leave at Lawrence Livermore Laboratory, Livermore, CA.

is referred to as the viscous interaction term and defined as:

$$V_{12} = \rho_1 F_{12}(u_2 - u_1) \tag{6}$$

where F_{12} is the inverse of the relaxation time for momentum transfer from phase 2 to phase 1. The momentum equation for the second phase is obtained by interchanging the subscripts in [4]–[6]. By adding the momentum equations for each phase, Soo (1976) obtains the mixture momentum equation in the form:

$$\frac{\partial}{\partial t}(\rho_m u_m) + \frac{\partial}{\partial x}(\rho_m u_m{}^2) = -\frac{\partial P}{\partial x} \tag{7}$$

or, by subtraction of the mixture continuity equation [2],

$$\rho_m \frac{\partial u_m}{\partial t} + \rho_m u_m \frac{\partial u_m}{\partial x} = -\frac{\partial P}{\partial x}. \tag{8}$$

For an application of these equations, consider the vapor–droplet flow field shown in figure 1. The droplets are injected at a steady rate to produce a one-dimensional, steady droplet-flow field. The evaporating droplets travel through the vapor and impact on the wall which continuously absorbs all the liquid, preventing any liquid accumulation on the wall, but which is impervious to the vapor phase. The vapor is free to flow out to the left past the droplet injection ports. The vapor is identified as phase 1 and the disperse phase (droplets) as phase 2. The suspension is assumed to be dilute; i.e. $\rho_2 \ll \rho_1$, according to Soo (1976).

The portion of the flow field under study lies between the two stations indicated on the figure. Station b is removed more than one droplet diameter away from the wall so that the droplet has not physically contacted the wall while passing the station. Thus there is no need to include a mechanical force on the droplet due to the wall in the droplet momentum equation.

Assuming Soo's equations are time-smoothed in the usual sense (Bird *et al.* 1960) and mass transfer occurs locally at a steady rate, the flow field between stations a and b is steady. The mixture continuity equation becomes

$$\rho_1 u_1 + \rho_2 u_2 = \rho_m u_m = \text{const.} \tag{9}$$

Because the wall is impervious to the vapor, u_1 must always be negative, so

$$\rho_m u_m < \rho_2 u_2 \tag{10}$$

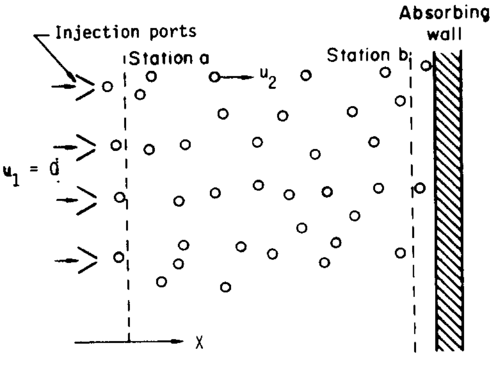

Figure 1. Schematic diagram of example flow field.

and

$$u_m < \frac{\rho_2}{\rho_1} u_2 . \qquad [11]$$

The maximum speed for u_1 is obtained by complete evaporation of the droplets before reaching station b which, upon evaluating the constant in [9] at the wall, leads to the following continuity equation:

$$\rho_1 u_1 + \rho_2 u_2 = 0 \qquad [12]$$

or

$$u_1 = -\frac{\rho_2}{\rho_1} u_2 . \qquad [13]$$

Because the mixture is dilute, one concludes

$$|u_1| \ll |u_2| . \qquad [14]$$

Substituting [10] and [11] into Soo's mixture momentum equation, [8], indicates

$$\frac{\mathrm{d}P}{\mathrm{d}x} \sim \rho_2 u_2 \frac{\rho_2}{\rho_1} \frac{\mathrm{d}u_2}{\mathrm{d}x} . \qquad [15]$$

For a very dilute suspension, the inertial interaction term for the disperse phase simplifies to

$$I_{21} \approx -(u_2 - u_1)\Gamma_2 + \frac{\mathrm{d}}{\mathrm{d}x}[\rho_2(u_1 - u_2)^2] \qquad [16]$$

which, for $|u_1| \ll |u_2|$, further reduces to

$$I_{21} \approx -u_2\Gamma_2 + \frac{\mathrm{d}}{\mathrm{d}x}(\rho_2 u_2^2) . \qquad [17]$$

Upon subtracting the continuity equation for phase 2,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho_2 u_2) = \Gamma_2 \qquad [18]$$

the inertial interaction term finally simplifies to

$$I_{21} \approx \rho_2 u_2 \frac{\mathrm{d}u_2}{\mathrm{d}x} . \qquad [19]$$

Substituting [15] and [19] into Soo's momentum equation for the disperse phase yields

$$\rho_2 u_2 \frac{\mathrm{d}u_2}{\mathrm{d}x} \approx \phi_2 \left[\frac{\rho_2}{\rho_1}\right] \rho_2 u_2 \frac{\mathrm{d}u_2}{\mathrm{d}x} + \rho_2 u_2 \frac{\mathrm{d}u_2}{\mathrm{d}x} + \rho_2 F_{21}(u_1 - u_2) . \qquad [20]$$

Realizing that the volume fraction occupied by the droplets is

$$\phi_2 < \frac{\rho_2}{\rho_1} \qquad [21]$$

and noting that the droplet acceleration and inertial interaction terms cancel, one concludes

$$F_{12}(u_1 - u_2) \approx 0\,. \qquad [22]$$

Because F_{12} cannot be zero, it must follow

$$u_1 \approx u_2\,. \qquad [23]$$

This result is inconsistent with [14].

The inconsistency in Soo's equations arises due to the presence of the inertial interaction terms in the phase equations. It is my contention that the correct form of the momentum equation for the disperse phase (Drew 1971, Crowe 1977), assuming locally uniform disperse-phase and uniform momentum efflux from the evaporating droplets, is

$$\rho_2\frac{\partial u_2}{\partial t} + \rho_2 u_2\frac{\partial u_2}{\partial x} = -\phi_2\frac{\partial P}{\partial x} + M + V \qquad [24]$$

where M is the "virtual-mass" term and V the viscous interaction force.† The virtual-mass term for a spherical droplet (Basset 1888) is given by

$$M = 0.5\rho_2\left[\frac{\bar{\rho}_1}{\bar{\rho}_2}\right]\frac{\mathrm{d}}{\mathrm{d}t}(u_1 - u_2) \qquad [25]$$

where $\bar{\rho}_1$ and $\bar{\rho}_2$ are material density of each phase (using Soo's terminology). The viscous interaction term, V, is defined as the force per unit volume of mixture as the result of viscous effects acting on the disperse phase.

The momentum equation for the disperse phase can be written in conservative form by multiplying the continuity equation for the disperse phase by u_2 and adding the result to [24], yielding:

$$\frac{\partial}{\partial t}(\rho_2 u_2) + \frac{\partial}{\partial x}(\rho_2 u_2^2) = -\phi_2\frac{\partial P}{\partial x} + M + V + u_2\Gamma_2\,. \qquad [26]$$

The corresponding momentum equation for the vapor phase is (Crowe 1977):

$$\frac{\partial}{\partial t}(\rho_1 u_1) + \frac{\partial}{\partial x}(\rho_1 u_1^2) = -\phi_1\frac{\partial P}{\partial x} - M - V + u_2\Gamma_1 \qquad [27]$$

where M and V are the same terms defined above.

Adding the momentum equation for each phase, [26] and [27], yields the mixture momentum equation:

$$\frac{\partial}{\partial t}(\rho_1 u_1 + \rho_2 u_2) + \frac{\partial}{\partial x}(\rho_1 u_1^2 + \rho_2 u_2^2) = -\frac{\partial P}{\partial x}\,. \qquad [28]$$

This equation may be written in terms of the mixture quantities as follows:

$$\frac{\partial}{\partial t}(\rho_m u_m) + \frac{\partial}{\partial x}(\rho_m u_m^2) + \frac{\partial}{\partial x}\left[\frac{\rho_1\rho_2}{\rho_1 + \rho_2}(u_1 - u_2)^2\right] = -\frac{\partial P}{\partial x} \qquad [29]$$

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†If the local velocity of the disperse phase is not uniform and one chooses to use the average disperse-phase velocity, an additional term appears (Drew 1971) which is similar to Soo's inertial interaction term but which is based on the average velocity of the disperse phase rather than that of the mixture.

which, using [2], becomes

$$\rho_m \frac{\partial u_m}{\partial t} + \rho_m u_m \frac{\partial u_m}{\partial x} + \frac{\partial}{\partial x}\left[\frac{\rho_1 \rho_2}{\rho_m}(u_1 - u_2)^2\right] = -\frac{\partial P}{\partial x}. \qquad [30]$$

Thus one concludes that a term similar to Soo's inertial interaction term should appear in the mixture momentum equation and not in the separate phase equations. Applying [30] to the vapor–droplet flow field discussed above leads to

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho_2 u_2^2) \approx -\frac{\mathrm{d}P}{\mathrm{d}x} \qquad [31]$$

which states that the loss of droplet momentum dur to aerodynamic drag is balanced by a pressure gradient.

The "inertial interaction" term in [30] is directly equivalent to the "apparent stress" arising from diffusion encountered in the momentum equations for a mixture of gaseous species (Penner 1957; Truesdell & Toupin 1960; Woods 1975). In the latter the "inertial interaction" term is frequently combined with the shear-stress tensor (Penner 1957). This practice can lead to significant difficulties in the case of disperse two-phase flows.

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REPLY TO PROFESSOR CROWE: "ON SOO'S EQUATIONS, etc."

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I am honored by Professor Crowe's attention to my work. It seems that his proving a fundamental relation via a specific example is contrary to the deductive nature of mechanics. However, some insight is actually gained when his example is solved correctly. Professor Crowe's finding appears to be a result of his liberal use of "≈'s".

Professor Crowe's present example could have many interesting conditions depending on